THE
MATHEMATICAL GAZETTE.

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THE REPORT OF THE COMMITTEE, AND INCOMMENSURABLES.

THE Report of the Committee of the Mathematical Association on the Teaching of Geometry having been published, I desire to draw attention to their treatment of the subject of ratio.

The Committee think (§ 48 of the Report) "*that an ordinary school course should not be required to include incommensurables*"; *but they do not provide any place for such a treatment of ratio as will form a preparation for that subject. I fear the result of this will be (though probably that was not intended) that the whole subject will be entirely ignored, not only in the case of those who are following an ordinary school course, but also in the case of those who wish to obtain accurate notions of the Infinitesimal Calculus; and of those who are preparing to take Honours in Mathematics at the Universities.*

Let me begin by endeavouring to clear away a misconception. The subject of ratio is an algebraic one, not a geometric one. Its treatment in the Fifth Book of Euclid has given rise to the idea that there is a geometric treatment of the subject as distinct from an algebraic one; and it would seem from the use of the word "algebraic" in § 47 (2) of the Committee's Report that they adopt this view.

Euclid employs a segment of a straight line to denote a magnitude, and more recently figures have been employed to illustrate what De Morgan has called the relative multiple scale of two magnitudes of the same kind. But these do not form an essential part of the reasoning. They are merely aids to the beginner in following the reasoning. All that Euclid assumes is that if A and B are two magnitudes of the same kind, and if r and s are any two integers, then it is possible to tell whether rA is greater than, equal to, or less than sB. This assumption seems to me to take the place of a definition of what is meant when we say that two magnitudes are of the same kind. However this may be, from this point onwards the whole of Euclid's work in the Fifth Book is strictly algebraic.

Leaving this point, it seems to me that the Report of the Committee would have given more help to teachers if they had sketched out an adequate "theory of measurement of lengths of lines and areas of rectangles for cases in which the lines and the sides of the rectangles are commensurable," § 47 (1), and if they had actually given the "algebraical treatment of of ratio and proportion for commensurables," § 47 (2), in a form which they

regard as sufficient. The teacher would then see how he was expected to explain the propositions

$$pA : qA = p : q \quad \text{[Euc. X. 5]},$$

and

$$rA : rB = A : B \quad \text{[Euc. V. 15]},$$

where A and B are magnitudes and p, q, r are integers. There is no difficulty about either of these propositions, but in the case of each of them there exists the danger that the beginner may consider himself entitled to treat the ratios as quotients of numbers containing a common factor, which may be removed.

In the elements of a subject it is necessary before all things that essential distinctions should be observed. There is nothing wanting in definiteness in Euclid's treatment of these subjects when expressed in a modern form, but it is very difficult to induce any one now to look at Euclid's Fifth Book, even though the cause for its difficulty has been discovered and removed. I doubt whether there exists any other treatment equally simple and equally clear. But the Committee have definitely rejected Euclid's line of argument, and it is therefore advisable to examine what they recommend.

I select (as an example of the results to which their line of treatment leads) § 55 of their Report, in which the Committee give their proof of the proposition: "If two triangles (parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of the rectangles contained by the sides about the equal angles."

The Committee say,

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC \,.\, AG}{\frac{1}{2}EF \,.\, DH} = \frac{BC \,.\, AG}{EF \,.\, DH},$$

and then, since

$$\frac{AG}{DH} = \frac{AB}{DE},$$

they say

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \,.\, AB}{EF \,.\, DE}.$$

I call attention to the omission of some intervening steps in the argument. It should be shown that

$$\frac{\text{rect. } BC \,.\, AG}{\text{rect. } EF \,.\, DH}$$

can be expressed as the product of the ratios $\frac{BC}{EF}$ and $\frac{AG}{DH}$; and then that the product of the ratios $\frac{BC}{EF}$ and $\frac{AB}{DE}$ can be expressed as

$$\frac{\text{rect. } BC \,.\, AB}{\text{rect. } EF \,.\, DE}.$$

How does the Committee justify the passage from the ratio of the areas of the rectangles to the product of the ratios of the sides, and conversely? I presume that it is implied that BC, AG, EF, DH are to be treated as commensurables and then the usual rules for the multiplication of rational fractions are to be applied. I cannot help feeling that the effect on the mind of the beginner will be that, after having had demonstrated to him certain rules for dealing with rational fractions, he has been told to apply these to other magnitudes which are not rational fractions. An undeveloped mind will probably accept this without demur, but its effect must be confusing. Possibly "the algebraic treatment of ratio and proportion for commensurables," which the Committee refer to, may be of such a nature that my objections may be met. But in the meanwhile I desire to suggest an alternative proof of the above proposition, and to indicate the way in which I think the whole of this part of the subject should be treated.

1. The fundamental problem in this part of the subject is to determine two straight lines having the same ratio as two equiangular parallelograms. Place the parallelograms so as to have a common angle, and the sides at the common angle in the same directions, then we have Fig. 1.

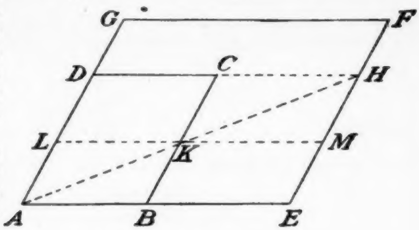


FIG. 1.

The two parallelograms are $ABCD$, $AEFG$. Produce DC to cut EF at H. Let AH cut CB at K. Through K draw a parallel to AE to cut AG at L, and EF at M. Then

$$ABCD = ABKL + LKCD = ABKL + BKME = AEML;$$

We have now constructed on AE, a side of the parallelogram $AEFG$, another parallelogram $AEML$ equal to $ABCD$ and equiangular with $AEFG$.

$$\therefore \frac{AEFG}{ABCD} = \frac{AEFG}{AEML} = \frac{AG}{AL}.$$

Now AG is a side of one of the parallelograms, and is therefore given. The line AL is found thus:

$$\frac{AE}{AB} = \frac{AH}{AK} = \frac{AD}{AL};$$

$$\therefore \frac{AE}{AB} = \frac{AD}{AL}.$$

Hence AL is the fourth proportional to the sides AE, AB, AD of the given parallelograms.

The length of AL depends only on the lengths of certain of the sides of the given parallelograms. *It does not depend at all on the value of the common angle.* If therefore we make that common angle a right angle and draw the rectangle whose sides are equal to AB, AD, and the rectangle whose sides are equal to AE, AG, we prove as above that

$$\frac{\text{rect. } AE \,.\, AG}{\text{rect. } AB \,.\, AD} = \frac{AG}{AL};$$

$$\therefore \frac{\text{parallelogram } AEFG}{\text{parallelogram } ABCD} = \frac{\text{rect. } AE \,.\, AG}{\text{rect. } AB \,.\, AD}.$$

2. The line AL can also be identified thus:

$$\because \frac{AE}{AB} = \frac{AD}{AL};$$

$$\therefore \text{rect. } AB \,.\, AD = \text{rect. } AE \,.\, AL.$$

Now the sides of the parallelogram $AEFG$ about A are AG, AE, whilst the sides of the parallelogram $ABCD$ about A are AB, AD. If we select one of the sides of the parallelogram $AEFG$, say AE, and on it describe a rectangle equal to the rectangle $AB \,.\, AD$, then the other side of it is the line AL; and the ratio of the parallelogram $AEFG$ to the parallelogram $ABCD$ is the same as that of the side of the parallelogram $AEFG$, which was not selected, viz. AG to the line AL which was found.

3. From Art. 1 it can be deduced that if two triangles have an angle equal, then their areas are proportional to the rectangles contained by the sides including the equal angles. That the areas of two triangles have the same ratio as the rectangles contained by their bases and altitudes, follows from the fact that the triangles are the halves of the rectangles.

4. The next step is to find two straight lines which are in the same ratio as the areas of two squares. This is included in Art. 1, but the result can be stated in a convenient form. Here we suppose $\hat{A}$ to be a right angle, $AE = AG$ and $AB = AD$. Then the equation of Art. 1

$$\frac{AE}{AB} = \frac{AD}{AL}$$

becomes

$$\frac{AE}{AB} = \frac{AB}{AL};$$

i.e. if AL be a third proportional to AE, AB, then

$$\frac{\text{the square on } AE}{\text{the square on } AB} = \frac{AE}{AL}.$$

5. The next step is to prove that the areas of similar triangles are proportional to the squares on corresponding sides.

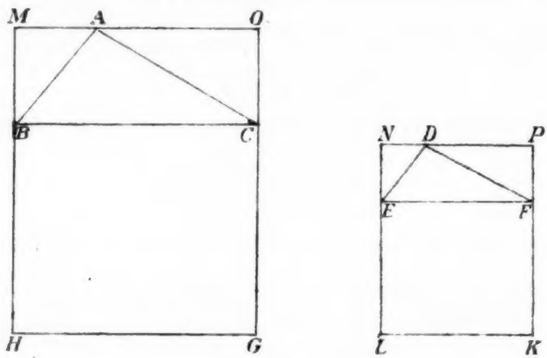


Fig. 2.

Let ABC, DEF (Fig. 2) be similar triangles, so that

$$B\hat{A}C = E\hat{D}F, \quad C\hat{B}A = F\hat{E}D, \quad A\hat{C}B = D\hat{F}E,$$

and

$$BC : EF = CA : FD = AB : DE.$$

On BC, EF describe the squares $BCGH$, $EFKL$. Also on BC describe the rectangle $BCOM$ which has a side passing through A, and on EF the rectangle $EFPN$ which has a side passing through D. We have to prove

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\text{square on } BC}{\text{square on } EF}.$$

But since

$$BCOM = 2\triangle ABC,$$

$$EFPN = 2\triangle DEF;$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BCOM}{EFPN}.$$

We have therefore to prove

$$\frac{BCOM}{EFPN} = \frac{BCGH}{EFKL},$$

$$i.e. \ \frac{BCOM}{BCGH} = \frac{EFPN}{EFKL},$$

but $$\frac{BCOM}{BCGH}=\frac{MB}{BH}=\frac{MB}{BC},$$

and $$\frac{EFPN}{EFKL}=\frac{NE}{EL}=\frac{NE}{EF}.$$

Hence it is necessary to prove

$$\frac{MB}{BC}=\frac{NE}{EF},$$

but $$\frac{BC}{BA}=\frac{EF}{ED}.$$

Hence it is necessary to prove

$$\frac{MB}{BA}=\frac{NE}{ED},$$

but the triangles MBA and NED have

$$B\hat{M}A=\text{rt. }\angle=E\hat{N}D,$$

$$M\hat{B}A=\text{rt. }\angle-A\hat{B}C=\text{rt. }\angle-D\hat{E}F=N\hat{E}D;$$

$$\therefore\ B\hat{A}M=E\hat{D}N.$$

Therefore the triangles MBA and NED are similar;

$$\therefore\ \frac{MB}{BA}=\frac{NE}{ED}.$$

$$\therefore\ \frac{\triangle ABC}{\triangle DEF}=\frac{\text{square on } BC}{\text{square on } EF}.$$

Hence the proposition is demonstrated.

6. The next step is to show that if A, B, C, D are 4 straight lines in proportion, then squares on A, B, C, D are also in proportion.

Find two other lines X and Y, such that

$$\frac{A}{B}=\frac{B}{X} \text{ and } \frac{C}{D}=\frac{D}{Y}.$$

Then since $\frac{A}{B}=\frac{C}{D}$, $\therefore\ \frac{B}{X}=\frac{D}{Y}$; $\therefore\ \frac{A}{X}=\frac{C}{Y}$.

But $$\frac{\text{square on } A}{\text{square on } B}=\frac{A}{X}, \quad \frac{\text{square on } C}{\text{square on } D}=\frac{C}{Y}, \text{ by Art. 4.}$$

$$\therefore\ \frac{\text{square on } A}{\text{square on } B}=\frac{\text{square on } C}{\text{square on } D}.$$

Conversely, if A, B, C, D be four straight lines, such that the squares on A, B, C, D are in proportion, then will A, B, C, D be in proportion.

Find a straight line E such that $\frac{A}{B}=\frac{C}{E}$;

$$\therefore\ \frac{\text{square on } A}{\text{square on } B}=\frac{\text{square on } C}{\text{square on } E};$$

but $$\frac{\text{square on } A}{\text{square on } B}=\frac{\text{square on } C}{\text{square on } D};$$

$$\therefore\ \frac{\text{square on } C}{\text{square on } D}=\frac{\text{square on } C}{\text{square on } E};$$

$$\therefore\ \text{square on } D=\text{square on } E.$$

From this it follows that $D=E$;

$$\therefore \frac{A}{B}=\frac{C}{D}.$$

Hence A, B, C, D are in proportion.

7. It remains only to prove that the areas of similar rectilineal figures are proportional to the squares on corresponding sides.

Let $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ be similar figures, so that $\hat{A}_1=\hat{A}_2$, $\hat{B}_1=\hat{B}_2$, $\hat{C}_1=\hat{C}_2$, $\hat{D}_1=\hat{D}_2$; and $A_1B_1 : A_2B_2=B_1C_1 : B_2C_2=C_1D_1 : C_2D_2=D_1A_1 : D_2A_2$.

To prove $$\frac{\text{area } A_1B_1C_1D_1}{\text{area } A_2B_2C_2D_2}=\frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}.$$

Let the figures be divided into the same number of similar triangles, as in Fig. 3 (Euc. VI. 20).

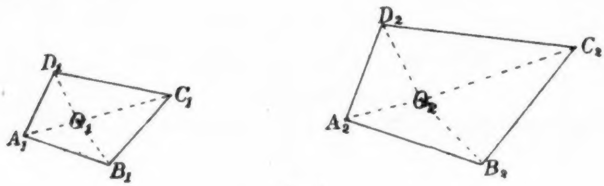


FIG. 3.

Then $$\frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}=\frac{\text{square on } A_1B_1}{\text{square on } A_2B_2};$$

$$\frac{\triangle B_1O_1C_1}{\triangle B_2O_2C_2}=\frac{\text{square on } B_1C_1}{\text{square on } B_2C_2}$$

$$=\frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}$$

$$=\frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}.$$

Similarly $\dfrac{\triangle C_1O_1D_1}{\triangle C_2O_2D_2}$ and $\dfrac{\triangle D_1O_1A_1}{\triangle D_2O_2A_2}$ each $=\dfrac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}$;

$$\therefore \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}=\frac{\triangle B_1O_1C_1}{\triangle B_2O_2C_2}=\frac{\triangle C_1O_1D_1}{\triangle C_2O_2D_2}=\frac{\triangle D_1O_1A_1}{\triangle D_2O_2A_2}.$$

Hence each of these ratios

$$=\frac{\triangle A_1O_1B_1+\triangle B_1O_1C_1+\triangle C_1O_1D_1+\triangle D_1O_1A_1}{\triangle A_2O_2B_2+\triangle B_2O_2C_2+\triangle C_2O_2D_2+\triangle D_2O_2A_2}$$

$$=\frac{\text{figure } A_1B_1C_1D_1}{\text{figure } A_2B_2C_2D_2};$$

$$\therefore \frac{\text{figure } A_1B_1C_1D_1}{\text{figure } A_2B_2C_2D_2}=\frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}$$

$$=\frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}.$$

Euc. VI. 22 follows from the above immediately.

In this way all necessity for using Euclid's phraseology involving the Compounding of Ratios and Duplicate Ratio is removed. At the same time I am not without hope that a reaction in favour of using the ideas of Euclid's Fifth Book may set in. There is this incidental advantage, that the frequent employment of what it is now usual to call Archimedes' Axiom (though Euclid used this axiom freely in his Fifth Book), which is involved in Euclid's line of treatment, draws attention to an idea of the greatest value. Its employment is involved in the proper treatment of magnitudes which exceed any given magnitude however great and magnitudes which are less than any given magnitude however small.

On this axiom for example depend

(*A*) The proofs of the algebraical propositions—

(1) that if $a > 1$: then it is possible to find n so great that a^n shall exceed any magnitude k however large; expressed usually $\underset{n=+\infty}{L} a^n = +\infty$;

*† (2) that if $0 < a < 1$; then it is possible to find n so great that a^n shall lie between 0 and ϵ, where ϵ is any positive magnitude however small; expressed usually $\underset{n=+\infty}{L} a^n = +0$.

(*B*) The proof of the geometrical proposition that points on a hyperbola exist whose distances from a focus exceed any length however great.

In this way the methods of the Fifth Book form an introduction to the Infinitesimal Calculus of the greatest value. M. J. M. Hill.

University College, London.

ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION.

The annual meeting of the Mathematical Association was held on Saturday, at King's College, London. In the absence of Mr. J. Fletcher Moulton, K.C., M.P., the retiring president, Professor A. Lodge was in the chair, and among those present were Professor A. R. Forsyth, Mr. F. W. Hill (treasurer), Mr. C. Pendlebury (secretary), Mr. A. W. Siddons, Mr. H. D. Ellis, Mr. W. H. Hudson, Mr. James Wilson, Mr. W. N. Roseveare, and Dr. F. S. Macaulay. The report of the council, which was adopted, stated that the association consisted of 351 members, and it referred to the work done by the committee appointed by the association to consider the subject of the teaching of elementary mathematics.

Professor Forsyth was elected president for the forthcoming year, and afterwards presided over the meeting. He remarked that the teaching of elementary mathematics had been a great deal under their consideration, and under the consideration of the local examination syndicate at Cambridge and that of the general syndicate of the whole University. It was not, however, for him to forecast what might be the issue of the deliberations upon the subject; but, as they probably knew, the local examination syndicate had modified their examinations and subsequent changes had been made in the schedule of mathematics. It was his privilege to serve on two of those bodies, and as a member of the committee that worked towards the modification of the regulations of the local examinations he wished to pay a tribute to the work done by the association.

* Although this proposition is necessary for the summation of an infinite number of terms of a Geometrical Progression with the common ratio less than unity, I believe there is only one English Text Book (Chrystal's *Algebra*) which contains a demonstration, and unfortunately that is far more difficult than it need be.

† Euclid proves this in the first proposition of the Tenth Book, for

$$0 < a \gtreqless \tfrac{1}{2}.$$

Mr. Siddons (Harrow School) submitted the report of the committee on the teaching of elementary mathematics, which, he said, had been criticised as very conservative. With regard to what had been done by various examining bodies, he remarked that in most cases their recommendations had been followed. University bodies were now alive to the necessity of not issuing regulations which clashed with one another. He thought it would be disastrous if that occurred, but he did not believe that there was any danger of that. The most immediate need was that the preparatory schools should move in the matter, and they should get the headmasters of such schools to adopt a more modern treatment of mathematics. It would not be done in the public schools unless the boys were taught from the beginning.

In a short discussion which followed the President said he regarded the teaching of the theory of incommensurables as a most advanced University subject, and the idea of attempting to teach the theory to the average schoolboy seemed to him to be almost an impossibility. He agreed with the line adopted by the committee in dealing with the report. It was desirable that they should not hurry changes. It did not lie with the public schools or the preparatory schools to make changes. They had a vast body of teachers in the small schools, but the great difficulty was to get at such teachers and induce them to change.

The report was adopted. Papers were afterwards read on "Some Class Diagrams for Intuitional Geometry" by Mr. E. M. Langley, on "The Representation of Imaginary Points on a Plane by Real Points" by Professor A. Lodge, and on "Incommensurables by Means of Continuous Decimals" by Mr. Edwin Budden.

MATHEMATICAL NOTES.

120. [V. 1. a.] *The Teaching of Arithmetic.*

The following answer, of which I have seen many in examination papers, is instructive.

Q.—Find to four places of decimals the value of

$$1+\frac{2}{x}+\frac{3}{x^2}+\frac{4}{x^3}+\text{etc. when } x=10.$$

A.—$1+\frac{2}{10}+\frac{3}{100}+\frac{4}{1000}+\frac{5}{10000}+\frac{6}{100000}$

$$=\frac{100000+20000+3000+400+50+6}{100000}$$

$$=\frac{123456}{100000}.$$

Ans.—123456.

$$\begin{array}{r} 100000\,)\,123456\,(\,123456 \\ \underline{100000} \\ 234560 \\ \underline{200000} \\ 345600 \\ \underline{300000} \\ 456000 \\ \underline{400000} \\ 560000 \\ \underline{500000} \\ 600000 \\ \underline{600000} \end{array}$$

G. H. Bryan.

121. [L.[1] 12. c.] *Isogonal Transformation.*

To prove 1 conic can be drawn through 5 pts. ....(1)

2 conics „ „ „ 4 pts. and touching 1 str. line. (2)

4 „ „ „ „ 3 pts. „ 2 str. lines. (3)

4 „ „ „ „ 2 pts. „ 3 „ (4)

2 „ „ „ „ 1 pt. „ 4 „ (5)

1 conic „ „ touching 5 str. lines. ....(6)

(1) Let A, B, C, D, E be the 5 pts. By isogonal transformation with regard to ABC as many conics can be drawn through A, B, C, D, E as str. lines through D', E' the isogonals of D, E, *i.e.* 1.

(2) Let A, B, C, D be the 4 pts. and q the str. line. Transform with regard to A, B, C. Then as many conics can be drawn to pass through A, B, C, D and touch q as str. lines can be drawn to pass through D' and touch conic q', the isogonals of D and q, *i.e.* 2.

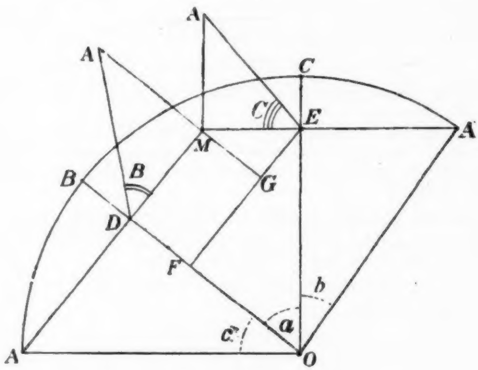
(3) Let A, B, C be the 3 given pts. and q, r the two str. lines. Transform with regard to A, B, C. Then as many conics can be drawn through A, B, C to touch q, r as str. lines can be drawn to touch conics q', r' the isogonals of q, r, *i.e.* 4.

(4), (5) and (6) follow from (3), (2), (1) by reciprocation.

H. L. TRACHTENBERG.

122. [K. 20. f.] *The Fundamental Formulae of Spherical Trigonometry.*

The proof, of which an outline note is subjoined, is a slight modification of one given by Mr. W. W. Lane, R.N. in his *Spherical Trigonometry.** The figure given below possesses the advantage of shewing all lines in their true lengths and angles of their true magnitude. Paper models obtained by cutting out $OABCAO$ or $OADAMAEAO$ and folding them up, are very easily made. The formulae for a right-angled spherical triangle are readily deduced in a similar manner.



Let ABC be a spherical triangle ; O the centre of the sphere.

Draw AD, AE, AM perpendicular to OB, OC and the plane BOC respectively.

Join MD, ME. Then
$$\begin{aligned} MD^2 &= AD^2 - AM^2 \\ &= OA^2 - OD^2 - (OA^2 - MO^2) \\ &= OM^2 - OD^2. \end{aligned}$$

*[These proofs are really due to our constant and most valued contributor, Mr. R. F. Davis, who discovered them in 1874, while an Undergraduate at Queen's College, Cambridge. Mr. Davis published them in the *Messenger of Mathematics*, Vol. IV., p. 102, and in the *Gazette*, Vol. I., p. 40. It is curious that such simple proofs have not found their way into general use. It is perhaps worth noting that they will be found in Sir Robert Ball's forthcoming *Spherical Astronomy*.

In 1878 Isaac Todhunter characterised the proofs as "interesting and elegant," and intended to introduce them into a new edition of his *Spherical Trigonometry*. J. Casey in 1888 wrote of "these beautiful proofs" that "they are certainly new to me, and I have no doubt they are original." He also expressed his intention to insert them in an Appendix to his *Spherical Trigonometry*. We are glad to have the opportunity of once more drawing attention to them.

W. J. G.]

Therefore MD is perpendicular to OB. Similarly ME is perpendicular to OC. Let the triangles ADO, AMD, AME, AEO revolve about OD, DM, ME and EO respectively, into the plane BOC. We thus obtain the annexed figure, and identify a, b, c, B, C therein.

Draw EF perpendicular to OD and MG perpendicular to EF.

We have $$OD=OF+FD, \text{ or, } R \text{ being the radius } OA,$$

$$R\cos c=R\cos b\cos a+R\sin b\cos C\sin a \qquad (FD=MG),$$

i.e. $$\cos c=\cos a\cos b+\sin a\sin b\cos C. \qquad (1)$$

Again $$EF=FG+GE, \text{ that is}$$

$$R\cos b\sin a=R\sin b\sin C\cot B+R\sin b\cos C\cos a;$$

or, dividing by $R\sin b$

$$\cot b\sin a=\cot B\sin C+\cos C\cos a. \qquad (2)$$

Again $$R\sin b\sin C=AM=R\sin c\sin B;$$

$$\therefore\ \frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}. \qquad (3)$$

The modification of the above figure when b or c is greater than 90° presents no difficulty.

C. S. JACKSON.

123. [L.[1] 1. a.] *Note on the Parabola.*

I. The general equation to the parabola must be of the form

$$(x-A)^2+(y-B)^2=(Lx+My+N)^2 \,|\, \{L^2+M^2\},$$

where A, B are the coordinates of the focus and $Lx+My+N=0$ is the equation of the directrix.

The above equation becomes by transposition $(Mx-Ly)^2+\ldots=0$, showing that the terms of the second degree form a perfect square and represent a line through the origin perpendicular to the directrix, that is, the diameter through the origin.

II. To reduce the general equation $(\alpha x+\beta y)^2+2gx+2fy+c=0$ to this form.

Assume $\beta x-\alpha y+\kappa=0$ as the equation to the directrix, where κ is at present undetermined and has to be found. Then

$$(\alpha^2+\beta^2)(x^2+y^2)+2x(g+\beta\kappa)+2y(f-\alpha\kappa)+c+\kappa^2=(\beta x-\alpha y+\kappa)^2,$$

and coordinates of focus are

$$-(g+\beta\kappa)\,|\,(\alpha^2+\beta^2),$$
$$-(f-\alpha\kappa)\,|\,(\alpha^2+\beta^2).$$

Also $$(g+\beta\kappa)^2+(f-\alpha\kappa)^2=(\alpha^2+\beta^2)(c+\kappa^2),$$

$$\kappa=\{f^2+g^2-c(\alpha^2+\beta^2)\}\,|\,2(\alpha f-\beta g).$$

Thus the focus and directrix are determined.

III. The axis is a line through the focus parallel to the diameter through the origin

$$\alpha\left(x+\frac{g+\beta\kappa}{\alpha^2+\beta^2}\right)+\beta\left(y+\frac{f-\alpha\kappa}{\alpha^2+\beta^2}\right)=0.$$

$$\alpha x+\beta y=(\alpha g+\beta f)\,|\,(\alpha^2+\beta^2).$$

IV. The latus rectum is twice the perpendicular from the focus on the directrix

$$=2\frac{-\beta(g+\beta\kappa)+\alpha(f-\alpha\kappa)+\kappa(\alpha^2+\beta^2)}{(\alpha^2+\beta^2)^{\frac{3}{2}}}=2\frac{\alpha f-\beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}}.$$

R. F. DAVIS.

REVIEWS.

Theorie der algebraischen Funktionen einer Variabeln und ihre Anwendung auf algebraische Kurven und Abelsche Integrale. By K. Hensel and G. Landsberg. Pp. xvi, 708. (Leipzig: Teubner.)

It is now more than twenty years since the publication in *Crelle's Journal* (vol. 92) of the memoir by Dedekind and Weber, in which it was shown how the theory of algebraic functions and their integrals could be based upon a strictly analytical foundation, without any appeal to geometrical intuition, or the premature introduction of transcendental functions. That paper practically concludes with the establishment of the Riemann-Roch theorem; and, so far as I am aware, the authors have never published the continuation of it which they intended. In any case, it is a great boon to have at last a treatise in which these ideas and methods are fully explained and developed.

The work is divided by the authors into six parts. Of these, the first deals with the mapping out of the values (u, z) which satisfy the equation $f(u, z)=0$, z being the independent variable. The Riemann sphere is employed, naturally, as a help to the reader; but the facts are established by the method of Puiseux, which is explained in a very thorough and lucid manner. It is also shown how the values of a function are connected by analytical continuation.

The second section deals with the fundamental properties of a corpus; norms, bases, etc., are defined, and, above all, the notion of prime divisors (in Dedekind's sense) is introduced. It is this, with the use of the corresponding notation and terminology, which is one of the most conspicuous features of the work. The notion is a difficult one, as it corresponds to that of an ideal primer in arithmetic: a prime divisor *may* be a function in the corpus, but it need not be. It is always, however, representable as the greatest common measure (in the proper sense) of two functions in the corpus; and this is perhaps the best way of thinking of it as an entity. To fully appreciate this notion of *divisor* requires time and patience; but when it has been grasped, its extreme value becomes manifest. In my opinion almost every page of this treatise, subsequent to the definition of divisor, bears witness to the precision and simplicity which is gained by adopting Dedekind's principles. The fact that the work is dedicated to Professor Dedekind tends to show that the authors are of the same opinion. Those of us who have been wondering for years why so great a genius has been so inadequately recognised may now hope for better things.

Section III. develops the theory of divisors, and terminates with the proof of the Riemann-Roch theorem. The fundamental problem of this part of the subject is the determination of all the divisors which are multiples of a given divisor. This is effected by the method of ideals. Another important problem is the determination, as a linear aggregate, or "Schaar," of all the integral divisors in a given class. This immediately leads us to the proof of the existence of p linearly independent Abelian differentials, and of the three normal types of Abelian integrals. The Riemann-Roch theorem follows very simply: in fact, almost as a corollary.

Section IV. deals with algebraic structures considered as curves. It is, in fact, an extremely valuable discussion of Plücker's equations in their strict and most general form. Besides this, we have an outline of residuation, and of the theory of adjoint curves.

Section V. treats of correspondence and transformation, and in particular of the "classes" of algebraic structures, and their moduli. Two structures are here taken as of the same class, when they define the same algebraic corpus.

Section VI. is on algebraic relations satisfied by Abelian integrals. This, it is hardly necessary to say, introduces the idea of periodicity, the theorems on interchange of argument and parameter, Abel's theorem, and the problem of inversion. With the statement of the last, the 37th lecture concludes. The 38th and last lecture gives a most interesting account of the historical development of the theory, and of the different ways in which it has been treated.

A reviewer's verdict on a work like this is very likely to be influenced by his own idea of the proper way of treating the subject. Personally, I prefer the method of Dedekind and Weber to any other that I have seen, and value this

treatise accordingly. But apart from the particular method of presentation, the book contains an immense amount of most interesting matter, expounded in a very clear and attractive way. To show that this is not merely my own impression, I may say that last term I lectured on the subject to a young graduate, following the lines of the book, and covering the first twenty lectures or so. The result was quite satisfactory; and the gentleman in question, of his own accord, expressed the opinion that the book was very clear.

I feel sure that others, as well as myself, will be very grateful to Drs. Hensel and Landsberg for providing them with such an excellent treatise on this important and fascinating subject. G. B. MATHEWS.

Dynamics of Rotation. An Elementary Introduction to Rigid Dynamics. By A. M. WORTHINGTON, C.B., M.A., F.R.S. (London: Longmans, Green & Co.) Fourth Edition. 1902. Pp. 164.

It is a misfortune of most courses on elementary dynamics that the problems which they include are in the majority of cases *unreal*. The "duffer" who cannot see what use it is to find the acceleration of a *perfectly smooth particle* on an inclined plane is certainly worthy of more consideration than the pet student who describes glibly how the acceleration of gravity may be found by Atwood's machine, using the formula $g(P-Q)/(P+Q)$, and finishing up with "Q.E.D.," when the method is fundamentally incorrect.

The dynamics of rotation of a rigid body ought to receive a more prominent place than has hitherto been accorded it, not only in the curriculum of students of physics and engineering, for which this book is specially written, but also in every course of mathematics. Many mathematicians have found considerable difficulty in beginning Dr. Routh's treatise for want of some grasp of the fundamental ideas involved in the subject, which become obscured when associated at the outset with too many higher analytical methods.

Prof. Worthington's book seems admirably adapted to give its readers a good insight into the principles of elementary rigid dynamics. There is no reason why the formulae for uniformly accelerated angular motion, $\omega=\omega_0+At$, $\theta=\omega_0t+\frac{1}{2}At^2$, and $\omega^2=\omega_0^2+2A\theta$, should not be taken immediately after, or concurrently with, the corresponding properties of linear motion, and if the time commonly spent in discussing elliptic motion about a focus along with the corresponding motion of the hodogram were expended on a study of this book much would be gained.

Whether the word *torque* is really a necessary innovation is a point about which opinions may differ. On the other hand, if the term is used generally, it is hardly as clear as it might be why in some places the author talks of a *couple*. Neither is it very clear how the fundamental property of moments, "Torques are found to be equal when the products of the force and the distance of its line of action from the axis are equal," may be "deduced from Newton's Laws of Motion" (p. 8). On the other hand the construction of "inertia skeletons" showing the moments of inertia of a body, and the positions of its principal axes, and the sections dealing with "centrifugal couples" are good features. The "slug" as unit of inertia may with advantage perish in common with the velo, celo, and tonal. Why is it that whenever it comes to teaching dynamics people will invent a lot of artificial units, and ignore the simple methods which beginners learn in connection with men mowing acres of grass or cats killing mice? G. H. BRYAN.

I Gruppi Continui di Trasformazioni. By ERNESTO PASCAL. Pp. xi., 358. Milan. (Manuali Hoepli.) 1903.

No idea is now more to the front in Pure Mathematics than that of a group. Discrete groups, and in particular groups of substitutions, pervade recent algebraical and arithmetical work: transformation groups are ever under consideration in continuous analysis and differential geometry. The conception of the latter class of groups, the subject-matter of the book before us, is the more recent. It eluded the grasp of the discoverers of facts of invariancy, for the thorough appreciation of which it was needed. Sophus Lie first formulated it in 1871; and to his genius is due a marvellously full development of the theory.

If we are given n relations expressing n letters x_1', x_2', ... x_n'—for brevity let us say n letters x'—as functions of n letters x and r parameters a, which are soluble without ambiguity for the letters x, we have a transformation, one between the x

and the x'. If we take another set of n relations, expressing n letters x'' as functions of the n letters x' and r parameters b, we have a second transformation, one between the x' and the x''. If between the two sets of relations we eliminate the letters x', we obtain n relations expressing the x'' as functions of the x and the two sets of parameters a, b—a transformation between the x and the x'', which, arising as it does from the sequence of the two former transformations, is called the product of the two in the order given. Order is not immaterial.

Now let the sets of n functional forms in the x, x' and x', x'' transformations be the same. The n functional forms in the resultant x, x'' transformation will, as a rule, differ from them. But there exist classes of cases in which the old functional forms are reproduced, with r functions of the parameters a, b for new parameters. In such a case the transformations, for all admissible values of the parameters, form a *group*, a *continuous* group as it is supposed that the parameters may be varied continuously.

This continuity, together with the group idea, suggests procedure in transformation by infinitesimal stages, and Lie makes the use of infinitesimal transformation fundamental in his researches. By its means the performance even of a finite transformation *of a group* is reduced to a matter of differential operation. An infinitesimal transformation gives to a function ϕ of the letters x an increment $\delta t X\phi$, where δt is infinitesimal and X is an operator linear in symbols of partial differentiation; and a finite transformation of the one-parameter group which the infinitesimal transformation specifies is performed by operation with

$$1+tX+\frac{t^2}{1.2}X^2+\ldots,$$

which may be written e^{tX}.

To those who like ourselves have been trained to expertness in continuous analysis this study should be inherently attractive. A certain difficulty which some of us have experienced in getting to feel at home in it, is perhaps due to the presence in our minds of an idea, as to what we should have meant by the performance of a transformation, which was not the idea dominant in Lie's mind. The latter idea has, of course, to take possession of us before confusion of thought as we follow Lie's arguments can cease. By performance of the x to x' transformation on a function we have to mean, not the substitution in the function for the letters x of their expressions in terms of the letters x', but the substitution for the letters x *of* the letters x', or, more frequently, of the expressions for these in terms of the letters x. Lie was before all things a geometer. Transformation to him meant, in the first place, rearrangement in space. An x to x' transformation meant transference of points with coordinates x to the places of other points with coordinates x' given as functions of the x.

Our danger of confusion is greatest when we deal with order in products of transformations. Taking formulae of successive transformation $x'=f(x)$, $x''=F(x')$, the passage from $\phi(x)$ to $\phi\{F[f(x)]\}$ is in Lie's theory a result of the first transformation followed by the second—a substitution first of x' for x and then of x'' for x'. The equally reasonable observation from a different point of view, that it would result from first replacing x by $F(x)$ according to the x'', x' relationship, and then replacing x by $f(x)$ according to the x', x relationship, is kept in the background. A lesson may however be drawn from this reversal of the order of ideas. We are in fact told that, if S denote the operation of transforming x to x' or $f(x)$, and T that of transforming x to $F(x)$, so that T' properly means that of transforming x' to $F(x')$ or x'', we must have

$$T'S=ST,$$

where, as in the book under review, the first performed operation in a product is written on the right. We shall encounter presently an example of extrication from temporary confusion by means of a case of this theorem.

The little book on Lie's theory which is before us deserves a hearty welcome. For a short time longer there is still no English book on the subject. Let those of us who know a little Italian peruse the present manual. It is all the easier to start upon because there is not room in it for the dignified style and the almost wearisome elaboration of the greater works brought out under Lie's own auspices. Few authors know so well as Sig. Pascal how to present higher mathematics in didactic form. The range of his mathematical learning is moreover cyclopedic.

The rate at which one useful and up-to-date *Manuale Hoepli* from his pen follows another is remarkable. Signs of haste in production, though not entirely absent, are rare.

The *Manuali Hoepli* are books of size for the pocket. Two pages would go on one of an ordinary octavo. The type, which is beautifully clear, is almost extravagantly large. The purchaser for half-a-crown of the present volume might well be forgiven for expecting only a meagre sketch of first principles, and not much of the analysis, perforce abounding in triple suffixes, etc., which is formidable of aspect even on the ample page of Teubner's *Lie-Engel.* He will be agreeably disappointed. The work is not unambitious. Its aim is "Without lack of rigour and generality to contain in little space all that forms the basis of this advanced part of pure mathematics." The author has at any rate succeeded in making clear in their complete forms the principles and processes of the general theory. Special theories, and in particular the whole subject of contact transformations, are reserved for a promised further volume.

Lie is of course followed closely, with judicious and lucid condensation. Essentials are only omitted from arguments in certain lengthy and elaborate investigations which are honestly presented as given in sketch only, or even, where fundamentals are not at stake, by mere reference. Expectation of more than Lie gives is at first raised by the mention of groups which are not "Gruppi di Lie," Lie having confined attention to groups which contain the identical transformation and of which all other transformations go in inverse pairs, whereas there may be groups which, for such reasons as that their parameters have not all values open to them, do not possess these properties. But groups other than Gruppi di Lie are not in the sequel subjected to analytical treatment.

The one important addition to Lie's analysis which occurs is a study (pp. 84, etc.) of the product of two distinct transformations each of which is a finite transformation of some one-parameter group, with matter leading up to the study, and some application of its conclusions. Proofs based on it are as a rule only given as alternative to Lie's proofs. In this study signs of haste are unfortunately noticeable, and the reader is hardly left convinced. It is not, however, now intended to throw doubt on the conclusions. There is support of which the author is not conscious. He is mistaken in believing that the matter was first dealt with by himself in 1901. He will find substantially the same theory and essentially the same conclusions in two papers, by Mr. J. E. Campbell "On a law of combination of operators bearing on the theory of continuous transformation groups," in Vols. XXVIII. and XXIX (1897) of the *Proceedings of the London Mathematical Society.*

An existence theorem is really in question in the study referred to. An x to x' transformation is performed by operation with e^{tX_1}, and then an x' to x'' transformation by operation with $e^{t'X_2'}$. X_2 is the formal result of replacing accented letters by unaccented in X_2'. The existence theorem is that a linear operator Z exists, and is linear with constant coefficients in tX_1, $t'X_2$, and a succession of alternants ("parentesi di Poisson") derived from these, which is such that the resultant x to x'' transformation can be performed by operation with e^{Z}, *i.e.* that

$$e^{t'X_2'}e^{tX_1}=e^{Z}.$$

That the theorem is one of existence is not stated by our author. It is even doubtful whether he has realized the fact. The existence of a linear Z, of some form or other, he appears to assume in the one word "naturalmente." It almost looks as if he took the operational equality just written to mean only that the product of two transformations is a transformation, whereas it really means that the product of two transformations each of which belongs to some group generated by an infinitesimal transformation is a third transformation with a like property. The proof of the existence of Z is left to depend on the actual discovery of the coefficients in a Z which suffices. This being so it is to be regretted that space could not be found for fuller proof of success in finding them.

There is another hasty misconception in the text, at the outset of the same investigation, which happily the author discovered in time to correct it in a note at the end of the book. The error was to suppose that in $e^{t'X_2'}e^{tX_1}$ we may without change of meaning replace X_2' by X_2. The correction is by means of a change of notation and a proof of what is practically the theorem of reversal of order

$$e^{t'X_2'}e^{tX_1}=e^{tX_1}e^{t'X_2},$$

which is a particular case of one arrived at just now. The immediate consequence of this

$$e^{t'X_2'} = e^{tX_1} e^{t'X_2} e^{-tX_1},$$

and the more general $$T' = STS^{-1},$$

are in interestingly close analogy with well-known facts as to the transformation of one substitution by another.

One of the many notes at the end of the volume contains a new proof, not given in full detail, of the difficult second part of Lie's third fundamental theorem.

E. B. ELLIOTT.

A Text-book of Field Astronomy for Engineers. By GEORGE C. COMSTOCK, Director of the Washburn Observatory, Professor of Astronomy in the University of Wisconsin. Pp. x, 202. (John Wiley & Sons, New York.)

This work consists of nine chapters, entitled: Introductory, Coordinates, Time, Corrections to coordinates, Rough determinations, Approximate determinations, Instruments, Accurate determinations, and The transit instrument. As its name implies, the book is intended for engineering students, and the "unconventional views" contained therein have developed during many years' experience in teaching the elements of practical astronomy. The "unconventional views" are excellent things. In the reviewer's experience the conventional view of a transit instrument from a student's stand-point is that three errors are connected with it; these he is quite eager to explain, but he is rather inclined to believe that the instrument is of no manner of use, and he is rather insulted if one asks him what it is for. Therefore, let those who teach and examine in astronomy, and who have no instruments to look after and no observations to make, get this book and acquire unconventional views as speedily as possible. The book will be found interesting and useful to the numerous amateurs who take pleasure in making time observations, etc. We have tried to show that the work will be useful and interesting to persons for whom it was not written, and we wish it to be inferred *a fortiori* that the engineer will find in it all he wants—and he often wants what he cannot find in the ordinary text-book on astronomy.

C. J. JOLY.

Specielle Algebraische und Transcendente Ebene Kurven. By GINO LORIA. Translated into German by FRITZ SCHUTTE. 2 vols., pp. xiv, 744, with 174 figures.

It may be admitted with considerable truth that few exercises are more useful to the student than the tracing of a number of curves. It is imperative that before the pupil is introduced to the general theory of curves he shall be familiarised with asymptotes, nodes, cusps, and the like, by the detailed discussion of numerical equations. The study of curves has its practical value in Mechanics and Physics, and we ought not to ignore its aesthetic value. Beauty and elegance, and even the "wild civility" of the asymmetric curves, have their attractions to the human young. Curves, moreover, appeal to the historic sense as perhaps does no other branch of science. The first real impulse which awakened the study of this department of Mathematics out of the lethargy from which it had suffered since the days of the Greeks was given by Descartes. Then came a period of development in which figure the names of Cavalieri, Wallis, Roberval, Pascal, and Newton. In reading the works of these founders of a new school, one has to be wary. Lack of communication, and the rarity and costliness of literature made it inevitable that there should be many working in the same field who were unknown to each other. Almost the last piece of work done by Pascal was the discussion of a curve to which he had given the name of the roulette. The same curve was dealt with by Roberval under the name of the trochoid; to this generation it is familiar under the title of the cycloid, and indeed it was known to Galileo under that name. The historical sense of the school-boy may in this case be tickled by the knowledge that Pascal was suffering from the toothache and insomnia when the thought of his "roulette" entered his mind. The disappearance of all aches and pains shortly after he had begun to brood over the subject was piously regarded by him as an intimation from above, that the Great Architect of the Universe was viewing with unqualified approval the attack of the problems which had exercised the ingenuity of Galileo when dealing with rolling curves in connection with the construction of the arches

of bridges. Most teachers may have noticed that it is comparatively easy to excite an interest in the personality of the inventor of a theorem or a law. Who would not prefer the name "Euler line" to "CONG line" for example? To study a curve in detail, to see how the various properties are brought to light, to realise how the history of the curve has been influenced by the progress of discoveries which perhaps had nothing to do with the art of curve tracing, to see where and why one man failed and his friend and rival, it may be, succeeded—this cannot but prove a stimulus to the young and enquiring mind, and may prove the germ of what in later years will ripen into efflorescence. But for even a short lecture such as is here suggested the teacher cannot rely for information on the ordinary text-books. Take for instance Frost's Curve Tracing. It contains, we verily believe, but two names in all its two hundred pages—Newton and De Gua. Think how much more interesting that admirable work might have been made by even the most elementary references to the history of the art. The teacher has no longer the excuse that he does not know whence to draw his material. This translation into German of Dr. Loria's encyclopaedic work will be far more than is necessary for such a modest programme as is permitted to the teacher when other claims are considered. It is monumental. From cover to cover it teems with interest. A copy should be on the shelves of every College Library, for "the sight of means to do *good* deeds make *good* deeds done." The student who wishes to carry further his researches will find all that he requires in the shape of references from the literature of the earliest days up to the most recent memoirs. The amount of patient labour which these two volumes with their 750 pages represent is colossal.

The Foundations of Geometry, by D. Hilbert. Authorised translation, by E. J. Townsend. Pp. viii, 142. 4s. 6d. net. (Open Court Co.: Kegan Paul.) 1902.

A translation of Hilbert's fascinating *Grundlagen der Geometrie* is heartily welcome in this country, and the volume under notice is further enriched by the author's additions, which appeared in the French translation which M. Laugel published some years ago (Gauthier-Villars). It also contains a summary of a memoir embodying Hilbert's latest researches, which has probably already appeared in the *Math. Ann.*

The first attempt to prove the concurrence in the plane of lines which are not parallel was made by Legendre. He showed that if any one triangle has the sum of its angles equal to two right angles, then the sum of the angles of all triangles will be two right angles; but he failed in his endeavour to prove the existence of one such triangle. Saccheri in 1733, and later Gauss, Lobatchewsky, and Bolyai attacked the same problem, but on different lines. They started with the negation of the axiom of parallels, and to the great surprise and alarm of Saccheri (v. Russell, *Foundations of Geometry*, p. 8) the result was more than one Geometry to the logical basis of which no objection could be found. Their success led to further investigations as to the axioms in general. The conception of space as a manifold of numbers gave Riemann, Helmholtz, and Lie the opportunity of establishing on an analytical basis both the non-Euclidean system of Lobatchewsky, and the system in which Euclid's "straight line" is avoided. In the former the sum of the angles of a triangle is always less, and in the latter always greater, than two right angles. On the other hand, we have the purely geometrical investigations of Veronese and Hilbert. How then are the researches of Hilbert to be placed with reference to the analytical researches of other workers in the same field? Helmholtz showed that Euclid's propositions were in disguise but the laws of motion of solid bodies. The non-Euclidean propositions were in the same manner the laws to which are subject bodies analogous to solid bodies, but with no physical existence. Lie went further. Combining all the possible transformations of a figure he calls the total a group. To each of these groups he attached a geometry; all these geometries have common properties; but the generality of his conclusions is impaired by the fact that all his groups are continuous. His space is a *Zahlenmannigfaltigkeit.* His geometries are subject to the forms of analysis and arithmetic. Now, as M. Poincaré points out (*Bull. des Sciences Mathématiques*, Sept. 1902), this is exactly where Hilbert comes in. His spaces are not *Zahlenmannigfaltigkeiten.* The objects he calls point, line, or plane are purely

logical conceptions. The most important of the additions to be found in the French translation are due to the investigations of Dr. Dehn, of which mention has been made already in our columns (No. 29, Oct. 1901, p. 94). The geometry which he constructs is one in which the sum of the angles of any triangle is two right angles; in which similar non-congruent triangles exist; and in which an infinite number of parallels to any straight line may be drawn through any point.

As a translation the volume before us cannot be said to be entirely successful. It has been unmercifully and somewhat undeservedly gibbeted by Prof. Halsted in *Science*, Aug. 22, 1902. A sober and detailed criticism by Dr. Hedrick of both this and the French translation will be found in the *Bull. of the American Math. Soc.*, Dec. 1902, to which considerations of space compel us to refer the reader, and in which will be found a long list of errors and misprints. We have carefully compared the French and English translations, and we find that Dr. Hedrick has omitted no point of any consequence, and that in our opinion his strictures are quite justified and necessary to the clear understanding of the text. With his list of errata the translation in question may be read by the student who can appeal to an expert for guidance, but on the whole we should prefer to place M. Laugel's book in his hands.

PROBLEMS.

469. [K. 20. b.] If e be so small that its cube and higher powers may be neglected, and $\phi-\theta+2e\sin\theta=\frac{3e^2}{4}\sin 2\theta$, then $\theta-\phi-2e\sin\phi=\frac{5e^2}{4}\sin 2\phi$.

ANON.

470. [D. 2. b.] If successive terms of a series are connected by the relation $u_nu_{n-3}-u_{n-1}u_{n-2}=2a$, and the first three terms are unity, prove that (i.) all the terms are integral functions of a, (ii.) any even term is the arithmetic mean of the adjacent terms; and find an expression for the nth term.

W. H. LAVERTY.

471. [K. 10. c.] Prove that the general form of the approximation to the length L of a circular arc when powers of $\frac{L}{R}$ beyond the $2(n+1)$th are neglected, R being the radius of the circle, is given by L_n where

$$L_na_1a_2a_3\dots a_n=l_0-l_1\frac{a_n}{a_1}2^{b_1}+l_2\frac{a_na_{n-1}}{a_1a_2}2^{b_2}-\dots+(-1)^nl_n2^{b_n},$$

where $a_r=1-4^r$; $b_r=r(r+2)$; $l_r=$chord of arc $\frac{L}{2^r}$.

(Huygens's approximation is given by $n=1$.)

R. M. MILNE.

472. [L.[1] 7. d.] Given the focus and directrix of a parabola, find by Euclidean methods the points in which the parabola cuts a given straight line. Solve also the same problem for the ellipse.

R. F. MUIRHEAD.

473. [L.[1] 17. e.] A family of conics have their axes along given lines and pass through a given point; show that the locus of the centre of curvature at the given point is a cubic whose asymptotes are all real.

C. F. SANDBERG.

474. [K. 2. e.] In a triangle ABC are inscribed three squares, of which the sides parallel to AB, BC, CA are DE', EF', FD'. Prove that the circles AEF', BFD', CDE' touch one another, and that the radius of the smaller circle which touches all three is $R/(4\cot\omega+7)$.

C. E. YOUNGMAN.

475. [K. 8. a.] The angles α, β, γ, δ of a quadrilateral satisfy the relation

$$\Sigma\cos^2\frac{\alpha}{2}+2\Pi\cos\frac{\alpha}{2}=2+2\Pi\sin\frac{\alpha}{2}.$$

Durham, 1902.

476. [B. 3. 4.] Eliminate θ, ϕ from the equations

$$\sin\theta+\sin\phi=a\,;\quad \sin 2\theta+\sin 2\phi=b\,;\quad \sin 3\theta+\sin 3\phi=c\,;$$

and eliminate θ from the equations

$$16\sin^5\theta-\sin 5\theta=a\,;\quad 16\cos^5\theta-\cos 5\theta=b. \qquad \text{(C.)}$$

477. [A. 2. b.] Solve the equations:

$$(b-c)x+(c-a)y+(a-b)z=0\,;$$
$$(b^2-c^2)yz+(c^2-a^2)zx+(a^2-b^2)xy=0\,;$$
$$(b-c)yz+(c-a)zx+(a-b)xy+(b-c)(c-a)(a-b)=0. \qquad \text{(C.)}$$

478. [A. 1. b.] If

$$\phi(x, y, n)\equiv\frac{x^n}{n}-x^{n-2}y+\frac{n-3}{\underline{|2}}x^{n-4}y^2$$
$$-\frac{(n-4)(n-5)}{\underline{|3}}x^{n-6}y^3+\frac{(n-5)(n-6)(n-7)}{\underline{|4}}x^{n-8}y^4-\dots,$$

where n is a positive integer, then shall

$$2\phi(\alpha, \beta, 2n)=\phi(\alpha^2-2\beta, \beta^2, n). \qquad \text{(C.)}$$

479. [K. 2. c.] Tangents are drawn to the circum- and 9-pt. circles of a triangle ABC where they are met by the join of A to the orthocentre. Find the area of the quadrilateral thus formed. (C.)

480. [L¹. 17. e.] Two equal and concentric ellipses are inscribed in a quadrilateral. Their axes are at an angle α. Shew that the area of the quadrilateral is

$$2\sqrt{(a^2-b^2)^2\sin^2\alpha+4a^2b^2}.$$

Two similar concentric ellipses of same eccentricity e, have their major axes at an angle α; shew how to find α if the foci of the first be on the second, and the ends of the minor axis of the second lie on the first. (C.)

SOLUTIONS.

406. [K. 3. a.] (Corrected.) *OAB is an isosceles triangle, base AB, and PAB is any other triangle on the same base and in the same plane: prove that*

$$4OA^2 . PA . PB . \cos^2\tfrac{1}{2}(AOB-APB)+BA^2 . OP^2=OA^2(PA+PB)^2. \qquad \text{(C.)}$$

Solution by R. N. Apte.

Put

$AB=c$; $OA=OB=a$; $PA=x$; $PB=y$; $PO=z$; $\angle APB=\theta$; $\angle APO=\phi$; while α, A denote the angles O, A of the triangle OAB.

(i.) Taking O, P on same side of P,

$$c^2=x^2+y^2-2xy\cos\theta,$$
$$a^2=x^2+z^2-2xz\cos\phi,$$
$$a^2=y^2+z^2-2yz\cos(\theta+\phi)\,;$$

$$\therefore\ ac\cos A=\tfrac{1}{2}(c^2+a^2-a^2)=-xy\cos\theta-xz\cos\phi+yz\cos(\theta+\phi)+x^2.$$

Also $$ac\sin A=xy\sin\theta+xz\sin\phi-yz\sin\overline{\theta+\phi}.$$

Rearranging, squaring, and adding, we find

$$z^2(x^2+y^2-2xy\cos\theta)$$
$$=a^2c^2+x^2y^2+x^4+2acxy\cos(A+\theta)-2acx^2\cos A-2x^3y\cos\theta,$$

i.e. $$c^2z^2=a^2c^2+c^2x^2+2acxy\cos(A+\theta)-2acx^2\cos A$$
$$=a^2c^2+2acxy\cos(A+\theta). \qquad (\because\ c=2a\cos A.)$$

(ii.) If O, P be on opposite sides of AB,

$$c^2z^2=a^2c^2+2acxy\cos(A-\theta).$$

$$\begin{aligned}\therefore\ c^2z^2&=a^2c^2+2acxy\cos(A\pm\theta)\\&=a^2(x^2+y^2-2xy\cos\theta)+2acxy\cos(A\pm\theta)\\&=a^2(x+y)^2-2xy[a^2(1+\cos\theta)-ac\cos(A\pm\theta)]\\&=a^2(x+y)^2-2a^2xy[1+\cos\theta-2\cos^2A\cos\theta\pm2\sin A\cos A\sin\theta]\\&=a^2(x+y)^2-2a^2xy[1-\cos(2A\pm\theta)]\\&=a^2(x+y)^2-2a^2xy[1+\cos(\alpha\mp\theta)]\qquad\left(\because\ A=\frac{\pi}{2}-\frac{\alpha}{2}.\right)\\&=a^2(x+y)^2-4a^2xy\cos^2\left(\frac{\alpha\mp\theta}{2}\right),\end{aligned}$$

according as O, P are on the same side or on opposite sides of AB.

Solution by R. M. MILNE.

Let C be mid-point of AB. Describe the circumcircle of the triangle APB, and denote the diameter which passes through O by TT'. A and B are the foci of an ellipse which passes through P, and PT, PT' are the tangent and normal at P. In the following analysis the constants a, b, c refer to this ellipse, and P is the point (x, y) referred to the usual axes. The point O will be (o, c).

Transposing, the question reads

$$4OA^2\,.\,PA\,.\,PB\cos^2\frac{AOB-APB}{2}=OA^2(PA+PB)^2-AB^2\,.\,OP^2.$$

$$\begin{aligned}\text{The Left-hand member }&=4\,.\,OA^2\,.\,PA\,.\,PB\cos^2(AOC+ATC)\\&=4\,.\,OA^2\,.\,PA\,.\,PB\frac{(OC\,.\,CT+AC^2)^2}{AO^2\,.\,AT^2}\\&=4\,.\,y\,.\,TT'\frac{\left(c\,.\,\frac{b^2}{y}+a^2e^2\right)^2}{TT'\,.\,\frac{b^2}{y}}\\&=\frac{4}{b^2}(cb^2+a^2e^2y)^2\\&=\frac{4a^2[c+e^2(y-c)]^2}{1-e^2}.\end{aligned}$$

$$\begin{aligned}\text{The Right-hand member}&=4a^2(c^2+a^2e^2)-4a^2e^2\,.\,OP^2\\&=4a^2[c^2+a^2e^2-e^2(x^2+\overline{y-c}^2)]\\&=4a^2[c^2+e^2(a^2-x^2)-e^2(y-c)^2]\\&=4a^2\left[c^2+\frac{e^2y^2}{1-e^2}-e^2(y-c)^2\right]\\&=\frac{4a^2}{1-e^2}[c^2(1-e^2)+e^2y^2-e^2(1-e^2)(y-c)^2]\\&=\frac{4a^2}{1-e^2}[c^2(1-e^2)^2+y^2e^4+2yce^2(1-e^2)]\\&=\frac{4a^2}{1-e^2}[c(1-e^2)+e^2y]^2\\&=\text{Left-hand member.}\end{aligned}$$

415. [K. 9. b.] *It is required to inscribe in a given circle $x^2+y^2=a^2$ a regular heptagon, one of whose angular points lies at (a, o). Prove that the two rectangular hyperbolas whose centres are at the points $(\frac{1}{4}a, \pm\frac{1}{4}a\sqrt{7})$, whose transverse axes are parallel to Ox, and which pass through the point (a, o), cut the circle also in the required angular points.* [*Suggestions for simpler constructions are invited.*] H. W. RICHMOND.

Solution by H. G. BELL.

The rectangular hyperbolas referred to are

$$\left(x-\frac{1}{4}a\right)^2-\left(y\pm\frac{\sqrt{7}}{4}a\right)^2=\frac{1}{8}a^2,$$

and these intersect $x^2+y^2=a^2$ in points whose abscissae are given by

$$x^2-\frac{1}{2}ax+\frac{1}{16}a^2-a^2+x^2\pm\frac{\sqrt{7}}{2}a\sqrt{a^2-x^2}-\frac{7}{16}a^2=\frac{1}{2}a^2,$$

$$(4x^2-ax-3a^2)^2=7a^2(a^2-x^2).$$

Dividing out by factor $x-a$, we get on reduction,

$$8x^3+4ax^2-4a^2x-a^3=0,$$

whose roots are

$$a\cos\frac{2\pi}{7},\quad a\cos\frac{4\pi}{7},\quad a\cos\frac{6\pi}{7}.$$

Hence the points of intersection give the vertices of inscribed heptagon.

Solution by R. F. DAVIS.

If in an auxiliary figure we construct the parabola $y^2=ax$, and the hyperbola $y(y-x+2a)=a^2$, the ordinate of the point of intersection of these curves nearest the origin is exactly the side of the regular polygon of fourteen sides inscribed in a circle of radius a.

For if λa be that ordinate, it will be found that

$$\lambda^3-\lambda^2-2\lambda+1=0\;;\quad\text{or}\quad \lambda=2\sin(\pi/14).$$

Solution by PROPOSER.

The rectangular hyperbola

$$(x-\tfrac{1}{4}a)^2-(y-\tfrac{1}{4}a\sqrt{7})^2=\tfrac{1}{8}a^2,$$

intersects the circle $x^2+y^2=a^2$ at (a, o), and at three other points. Write the equation of the hyperbola

$$\{x+iy-\tfrac{1}{4}a(1+i\sqrt{7})\}^2+\{x-iy-\tfrac{1}{4}a(1-i\sqrt{7})\}^2=\tfrac{1}{4}a^2,$$

and substitute $x=a\cos\theta$, $y=a\sin\theta$, $\cos\theta+i\sin\theta=t$; then

$$\{t-\tfrac{1}{4}(1+i\sqrt{7})\}^2+\{t^{-1}-\tfrac{1}{4}(1-i\sqrt{7})\}^2=\tfrac{1}{4}\;;$$

$$t^2+t^{-2}-\tfrac{1}{2}(t+t^{-1})-\tfrac{1}{2}i\sqrt{7}(t-t^{-1})-1=0\;;$$

either $\qquad t=1\quad\text{or}\quad t^3+\tfrac{1}{2}t^2-\tfrac{1}{2}t-1-\tfrac{1}{2}i\sqrt{7}(t^2+t)=0.$

Now if

$$p=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7},\quad p^7=1,\quad\text{and}\quad p^6+p^5+p^4+p^3+p^2+p=-1.$$

$$(p+p^2+p^4)+(p^6+p^5+p^3)=-1,$$

$$(p+p^2+p^4)\times(p^6+p^5+p^3)=3p^7+p^6+p^5+p^4+p^3+p^2+p=2.$$

Thus $p+p^2+p^4$ and $p^6+p^5+p^3$ are roots of $z^2+z+2=0$, and

$$p+p^2+p^4=\tfrac{1}{2}(-1+i\sqrt{7}),\quad p^6+p^5+p^3=\tfrac{1}{2}(-1-i\sqrt{7}).$$

Hence the equation whose roots are p, p^2, p^4 is

$$(t-p)(t-p^2)(t-p^4)=0,$$

or

$$t^3-t^2(p+p^2+p^4)+t(p^6+p^5+p^3)-1=0,$$

or

$$t^3+\tfrac{1}{2}t^2-\tfrac{1}{2}t-1-\tfrac{1}{2}i\sqrt{7}(t^2+t)=0.$$

$\therefore$ The hyperbola cuts the circle in four points whose vectorial angles measured from Ox are

$$0,\ \frac{2\pi}{7},\ \frac{4\pi}{7},\ \frac{8\pi}{7}.$$

416. [M[1]. 2. a.] *Find on a given straight line two points which shall be homologous in two given similar figures.* C. E. YOUNGMAN.

Solution by R. F. DAVIS.

Let P, Q be homologous points in the two figures; O the centre of similitude. Upon the given straight line AB take points X, Y, such that the angle $OXB=OPQ$ and $OYA=OQP$. Then the triangles OPQ, OXY are similar; etc.

419. [R. 9. b.] *A, B are two smooth holes in a smooth horizontal table, distance $2a$ apart; a particle of mass M rests on the table midway between A and B, and a particle of mass m hangs beneath the table suspended from M by two equal weightless inextensible strings, each of length $a(1+\sec\alpha)$, passing through A and B; a blow J is given to M in a direction perpendicular to AB. Shew that if $J^2>2Magm\tan\alpha$, M will oscillate to and fro through a distance $2a\tan\alpha$, but if J^2 is less than this quantity and $=2Mmag(\tan\alpha-\tan\beta)$, the distance through which M oscillates will be $2a\sqrt{(\sec\alpha-\sec\beta)(\sec\alpha-\sec\beta+2)}$.*

St. John's (C.), 1895.

Solution by W. F. BEARD; R. F. DAVIS; C. V. DURELL.

Let θ be the angle (in the plane of the table) which the upper portion of either string makes at any instant with AB; ϕ, the angle in the vertical plane which the lower portion makes with AB. Initially $\phi=\alpha$, and the energy of the system is $J^2/2M$. The potential energy when the system is instantaneously at rest is $mga(\tan\alpha-\tan\phi)$, and the kinetic energy $=0$. Hence

$$mga(\tan\alpha-\tan\phi)=J^2/2M.$$

(i.) If $$J^2=2Mmga(\tan\alpha-\tan\beta),$$

$\phi=\beta$, $$\sec\theta=1+\sec\alpha-\sec\beta\ \text{[from length of string]},$$

and $$\tan^2\theta=(\sec\alpha-\sec\beta)^2+2(\sec\alpha-\sec\beta),$$

so that M oscillates through a distance

$$2a\sqrt{(\sec\alpha-\sec\beta)(\sec\alpha-\sec\beta+2)}.$$

(ii.) If $J^2=2Mmga\tan\alpha$, $\phi=0$, and the lower particle comes to rest in the line AB, and M will oscillate through a distance $2a\tan\alpha$.

(iii.) If $J^2>2Mmga\tan\alpha$, m strikes the bottom of the table with a finite velocity, and M still oscillates through $2a\tan\alpha$, but the oscillations are not periodic until after an infinite number of impacts unless the table be inelastic.

425. [K. 2. c.] *The sides of a triangle ABC are bisected in D, E, F; shew that on the circle DEF four positions of a point P may be found such that*

$$DP \pm EP \pm FP = 0,$$

and that these are the four points where the circle is touched by the incircle and excircles of the triangle ABC.

Shew also that the tangents to the circle at these four points are also tangents of the ellipse which touches each side of ABC at its middle point.

H. W. RICHMOND.

Solution by PROPOSER.

If D, E, F be three points on a circle touched by a second circle at P, and DP, EP, FP cut the second circle in D', E', F',

$$DD' : EE' : FF' :: DP : EP : FP.$$

$\therefore$ the tangents from D, E, F to the second circle have their lengths, viz.,

$$\{DD'.DP\}^{\frac{1}{2}}, \quad \{EE'.EP\}^{\frac{1}{2}}, \quad \{FF'.FP\}^{\frac{1}{2}},$$

proportional to DP, EP, FP.

Hence if P be the point of contact of the incircle of ABC and the N.P. circle

$$DP : EP : FP :: \tfrac{1}{2}(b \sim c) : \tfrac{1}{2}(a \sim c) : \tfrac{1}{2}(a \sim b),$$

for the last are the lengths of the tangents from D, E, F to the incircle.

$$\therefore \; DP \pm EP \pm FP = 0.$$

So for the escribed circles.

Again, if p_1, p_2, p_3 be the perpendiculars from D, E, F on the tangent at P,

$$p_1 : p_2 : p_3 :: DP^2 : EP^2 : FP^2.$$

$$\therefore \; \sqrt{p_1} \pm \sqrt{p_2} \pm \sqrt{p_3} = 0.$$

Hence the tangent touches a conic through D, E, F, the tangents at D, E, F being sides of ABC.

450. [K. 20. e.] *In any triangle, r_1, r_2, r_3 being the ex-radii, and s the semi-perimeter, shew that*

$$\frac{r_2+r_3}{(s-a)\sin A} = \frac{r_3+r_1}{(s-b)\sin B} = \frac{r_1+r_2}{(s-c)\sin C},$$

and give a symmetrical expression for the common value. E. N. BARISIEN.

Solution by W. F. BEARD; J. M. CHILD.

$$\frac{r_2+r_3}{(s-a)\sin A} = \frac{bc}{2(s-a)} \cdot \frac{r_2+r_3}{\Delta}$$

$$= \frac{bc}{2(s-a)}\left(\frac{1}{s-b} + \frac{1}{s-c}\right)$$

$$= \frac{abc}{2(s-a)(s-b)(s-c)}$$

$$= \frac{abcs}{2\Delta^2}.$$

451. [J. 1. a. α.] *If $f(n)$ be the number of permutations of n letters altogether, on condition that no letter is moved more than one place from its original position in a given order, then*

$$L_{n=\infty} \frac{f(n)}{f(n+1)} = 2 \sin 18°.$$ E. M. LANGLEY.

Solution by W. F. BEARD.

The number of permutations in which the first letter retains its place is $f(n-1)$: the number in which it is moved is $f(n-2)$,

$$\therefore\ f(n)=f(n-1)+f(n-2).$$

Also $f(1)=1$, $f(2)=2$; and if

$$(1-x-x^2)=(1-\alpha x)(1-\beta x),$$

α being positive and numerically greater than β, so that

$$\alpha+\beta=1 : \alpha\beta=-1,$$

$f(n)$ is the coefficient of x^n in the expansion in ascending powers of x of

$$\frac{x(1-x)}{1-x-x^2}=\frac{x}{\alpha-\beta}\left\{\frac{\alpha-1}{1-\alpha x}-\frac{\beta-1}{1-\beta x}\right\};$$

$$\therefore\ \frac{f(n)}{f(n+1)}=\frac{(\alpha-1)\alpha^{n-1}-(\beta-1)\beta^{n-1}}{(\alpha-1)\alpha^n-(\beta-1)\beta^n}=\frac{-\beta\alpha^{n-1}+\alpha\beta^{n-1}}{-\beta\alpha^n+\alpha\beta^n}$$

$$=\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha^{n-1}-\beta^{n-1}}=\frac{1}{\alpha}\,\frac{1-\left(\frac{\beta}{\alpha}\right)^{n-2}}{1-\left(\frac{\beta}{\alpha}\right)^{n-1}};$$

$$\therefore\ Lt_{n=\infty}\frac{f(n)}{f(n+1)}=\frac{1}{\alpha}\qquad\left[\because\ \left|\frac{\beta}{\alpha}\right|<1\right]$$

$$=\frac{-1+\sqrt{5}}{2}=2\sin 18^\circ.$$

Solution by J. M. CHILD.

If $f(n)=r+s$, where r of the permutations end in a_n, and s in a_{n-1}, then

$$f(n+1)=2r+s,$$

of which r end in a_n, and $r+s$ in a_{n+1}, and

$$f(n+2)=2(r+s)+r=f(n+1)+f(n).$$

Hence $\dfrac{f(n)}{f(n+1)}$ is in general the reciprocal of the convergents to the continued fraction

$$1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}.\qquad\left[=x=1+\frac{1}{x}.\right]$$

Hence $$L_{n=\infty}\frac{f(n)}{f(n+1)}=\frac{1}{x}=\frac{1}{\frac{\sqrt{5}+1}{2}}=\frac{\sqrt{5}-1}{2}=2\sin 18^\circ.$$

453. [L^1. 17. e.] *Two conics have a common focus and their directrices are at right angles: what is the condition that their common chord may pass through the common focus?* A. F. VAN DER HEYDEN (Durham, 1902).

Solution by W. F. BEARD.

Let S be the common focus; PQ the common chord; R the intersection of the directrices; e, e' the eccentricities; PM, PM', QN, QN' perpendiculars on the directrices.

$$e\,.\,PM=SP=e'\,.\,PM' : e\,.\,QN=e'\,.\,QN';$$

$$\therefore\ PQ \text{ passes through } R,$$

and if S lies on PQ, $\qquad e\,.\,SX=e'\,.\,SX',$

i.e. the conics have equal latera recta ; this is also easily proved from the polar equations of the conics.

[The semi latus rectum is a harmonic mean between the segments of any focal chord ; hence if two conics have a common focus and a common chord passing through that focus, they must have equal latera recta : the directrices need not be at right angles.]

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ERRATA.

p. 248	line	12 up.	*For*	"from"	*read*	"form."
p. 248	,,	10	,,	"$\frac{1}{2}$"	,,	"$\frac{1}{3}$."
p. 248	,,	12	*Omit*	"again."		
p. 249	,,	29	*For*	"Goldback"	*read*	"Goldbach."
p. 249	,,	30	,,	"integer"	,,	"even number."

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